RECURSION FORMULAE OF HIGHER WEIL-PETERSSON VOLUMES

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Abstract. In this paper we study effective recursion formulae for computing intersection numbers of mixed ψ and κ classes on moduli spaces of curves. By using the celebrated Witten-Kontsevich theorem, we generalize Mulase-Safnuk form of Mirzakhani's recursion and prove a recursion formula of higher Weil-Petersson volumes. We also present recursion formulae to compute intersection pairings in the tautological rings of moduli spaces of curves.

1. Introduction

We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus g complex algebraic curves. We have the morphism that forgets the last marked point,

$$\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \ldots, \sigma_n$ the canonical sections of π, and by $D_1, \ldots, D_n$ the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf, we have the following tautological classes on moduli spaces of curves.

$$\begin{aligned}
\psi_i &= c_1(\sigma_i^*(\omega_\pi)) \\
\kappa_i &= \pi_* \left(c_1 \left(\omega_\pi \left(\sum D_i \right) \right)^{i+1} \right) \\
\lambda_l &= c_l(\pi_*(\omega_\pi)), \quad 1 \le l \le g.
\end{aligned}$$

The classes κ_i were first defined by Mumford [21] on $\overline{\mathcal{M}}_g$. Their generalization to $\overline{\mathcal{M}}_{g,n}$ here is due to Arbarello-Cornalba [1, 2]. Before that time, the classes κ_i were defined as $\pi_*(c_1(\omega_\pi)^{i+1})$. Arbarello-Cornalba's definition turned out to be the correct one especially from the point of view of the restrictions to the boundary strata.

We are interested in the following intersection numbers

$$\langle \kappa_{b_1} \cdots \kappa_{b_k} \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \kappa_{b_1} \cdots \kappa_{b_k} \psi_1^{d_1} \cdots \psi_n^{d_n},$$

where $\sum b_j + \sum d_j = 3g - 3 + n$. When $d_1 = \cdots = d_n = 0$, these intersection numbers are called the higher Weil-Petersson volumes of moduli spaces of curves.

The fact that intersection numbers involving both κ classes and ψ classes can be reduced to intersection numbers involving only ψ classes was already known to Witten [9], and has been developed by Arbarello-Cornalba [1], Faber [7] and Kaufmann-Manin-Zagier [13] into a beautiful combinatorial formalism. Faber has a wonderful maple program computing these intersection numbers.

In a series of innovative papers [18, 19], Mirzakhani obtained a beautiful recursion formula of the Weil-Petersson volumes of the moduli spaces of bordered Riemann surfaces. As discussed by Mulase and Safnuk in [20, 23], Mirzakhani's recursion formula is equivalent to the following enlightening recursion relation of intersection numbers.

$$(2k_1+1)!! \langle \kappa_1^{k_0} \tau_{k_1} \cdots \tau_{k_n} \rangle_g$$

$$
\begin{aligned}
&= \sum_{j=2}^{n}\sum_{l=0}^{k_0} \frac{k_0!}{(k_0-l)!}\frac{(2(l+k_1+k_j)-1)!!}{(2k_j-1)!!}\beta_l\langle\kappa_1^{k_0-l}\tau_{k_1+k_j+l-1}\prod_{i\neq 1,j}\tau_{k_i}\rangle_g \\
&+\frac{1}{2}\sum_{l=0}^{k}\sum_{d_1+d_2=l+k_1-2}\frac{k_0!}{(k_0-l)!}(2d_1+1)!!(2d_2+1)!!\beta_l\langle\kappa_1^{k_0-l}\tau_{d_1}\tau_{d_2}\prod_{i\neq 1}\tau_{k_i}\rangle_{g-1} \\
&+\frac{1}{2}\sum_{\substack{m_0+n_0=k_0-l\\ I\coprod J=\{2,\dots,n\}}}\sum_{l=0}^{k_0}\sum_{d_1+d_2=l+k_1-2}\frac{k_0!}{m_0!n_0!}(2d_1+1)!!(2d_2+1)!!\beta_l \\
&\qquad\times\langle\kappa_1^{m_0}\tau_{d_1}\prod_{i\in I}\tau_{k_i}\rangle_{g'}\langle\kappa_1^{n_0}\tau_{d_2}\prod_{i\in J}\tau_{k_i}\rangle_{g-g'},
\end{aligned}
$$

where

$$\beta_l=(2^{2l+1}-4)\frac{\zeta(2l)}{(2\pi^2)^l}=(-1)^{l-1}2^l(2^{2l}-2)\frac{B_{2l}}{(2l)!}.$$

In a previous paper [16], it is shown that the Witten-Kontsevich theorem implies the Mulase-Safnuk form of Mirzakhani's recursion formula. Its relationship with matrix integrals has been studied by Eynard and Orantin [5, 6].

More discussions about computations of Weil-Petersson or higher Weil-Petersson volumes can be found in the papers [10, 12, 13, 17, 22, 24, 26, 27].

Now we fix notation as in [13]. Consider the semigroup N^∞ of sequences $\mathbf{m}=(m(1),m(2),\dots)$ where $m(i)$ are nonnegative integers and $m(i)=0$ for sufficiently large i. Denote by $\boldsymbol{\delta}_a$ the sequence with 1 at the a-th place and zeros elsewhere.

Let $\mathbf{m},\mathbf{t},\mathbf{a_1},\dots,\mathbf{a_n}\in N^\infty$, $\mathbf{m}=\sum_{i=1}^n\mathbf{a_i}$, and $\mathbf{s}:=(s_1,s_2,\dots)$ be a family of independent formal variables.

$$|\mathbf{m}|:=\sum_{i\geq 1}im(i),\quad ||\mathbf{m}||:=\sum_{i\geq 1}m(i),\quad \mathbf{s}^{\mathbf{m}}:=\prod_{i\geq 1}s_i^{m(i)},\quad \mathbf{m}!:=\prod_{i\geq 1}m(i)!,$$

$$\binom{\mathbf{m}}{\mathbf{t}}:=\prod_{i\geq 1}\binom{m(i)}{t(i)},\quad \binom{\mathbf{m}}{\mathbf{a_1},\dots,\mathbf{a_n}}:=\prod_{i\geq 1}\binom{m(i)}{a_1(i),\dots,a_n(i)}.$$

Let $\mathbf{b}\in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}):=\prod_{i\geq 1}\kappa_i^{b(i)}.$$

Theorem 1.1. *Let $\mathbf{b}\in N^\infty$ and $d_j\geq 0$. Then*

$$
\begin{aligned}
(1)\quad &\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\frac{(2d_1+2|\mathbf{L}|+1)!!}{(2|\mathbf{L}|+1)!!}\langle\kappa(\mathbf{L}')\tau_{d_1+|\mathbf{L}|}\prod_{j=2}^{n}\tau_{d_j}\rangle_g \\
&=\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\kappa(\mathbf{b})\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g \\
&+\frac{1}{2}\sum_{r+s=|d_1|-2}(2r+1)!!(2s+1)!!\langle\kappa(\mathbf{b})\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1} \\
&+\frac{1}{2}\sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b}\\ I\coprod J=\{2,\dots,n\}}}\sum_{r+s=d_1-2}\binom{\mathbf{b}}{\mathbf{e}}(2r+1)!!(2s+1)!! \\
&\qquad\times\langle\kappa(\mathbf{e})\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa(\mathbf{f})\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}
$$

Theorem 1.2. *Let* $\mathbf{b} \in N^\infty$ *and* $d_j \geq 0$. *Then*

$$
\begin{aligned}
(2)\quad & (2d_1+1)!!\langle \kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g \\
&= \sum_{j=2}^{n} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!} \langle \kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i}\rangle_g \\
&+ \frac{1}{2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{L}')\tau_r\tau_s \prod_{i=2}^{n} \tau_{d_i}\rangle_{g-1} \\
&\quad + \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I\coprod J=\{2,\ldots,n\}}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}} (2r+1)!!(2s+1)!! \\
&\qquad\qquad \times \langle \kappa(\mathbf{e})\tau_r \prod_{i\in I}\tau_{d_i}\rangle_{g'} \langle \kappa(\mathbf{f})\tau_s \prod_{i\in J}\tau_{d_i}\rangle_{g-g'},
\end{aligned}
$$

where the constants $\alpha_{\mathbf{L}}$ *are determined recursively from the following formula*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!} = 0, \qquad \mathbf{b}\neq 0,$$

namely

$$\alpha_{\mathbf{b}} = \mathbf{b}! \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}'\neq \mathbf{0}}} \frac{(-1)^{||\mathbf{L}'||-1}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}, \qquad \mathbf{b}\neq 0,$$

with the initial value $\alpha_{\mathbf{0}} = 1$.

Denote $\alpha(l,0,0,\dots)$ by α_l, we recover Mirzakhani's recursion formula with

$$\alpha_l = l!\beta_l = (-1)^{l-1}(2^{2l}-2)\frac{B_{2l}}{(2l-1)!!}.$$

We also have

$$\alpha(\boldsymbol{\delta}_l) = \frac{1}{(2l+1)!!}.$$

Setting $\mathbf{b} = \mathbf{0}$, we get the Witten-Kontsevich theorem [25, 14] in the form of DVV recursion relation [4].

Note that Theorems 1.1 and 1.2 hold only for $n \geq 1$. If $n = 0$, i.e. for higher Weil-Petersson volumes of $\overline{\mathcal{M}}_g$, we may apply the following formula first (see Proposition 3.1).

$$\langle \kappa(\mathbf{b})\rangle_g = \frac{1}{2g-2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1}\kappa(\mathbf{L}')\rangle_g. \tag{3}$$

So we can use Theorems 1.1 and 1.2 to compute any intersection numbers of ψ and κ classes recursively with the three initial values

$$\langle\tau_0\kappa_1\rangle_1 = \frac{1}{24}, \qquad \langle\tau_0^3\rangle_0 = 1, \qquad \langle\tau_1\rangle_1 = \frac{1}{24}.$$

We have computed a table of $\alpha_{\mathbf{L}}$ for all $|\mathbf{L}| \leq 15$ and have written a maple program [28] implementing Theorems 1.1 and 1.2.

In the arguments of Mirzakhani, Mulase and Safnuk, they use Wolpert's formula [26]

$$\kappa_1 = \frac{1}{2\pi^2}\omega_{WP},$$

where ω_{WP} is the Weil-Petersson Kähler form. Since Wolpert's formula has no counterpart for higher degree κ classes, there is no a priori reason that Theorem 1.2 shall be true.

We are led to Theorem 1.2 also by the discovery that ψ and κ classes are compatible, namely recursions of pure ψ classes can be neatly generalized to recursions including both ψ and κ classes, where κ_1 plays no special role. This fact is equivalent to a relation of generating functions in Theorem 4.4.

For $\mathbf{b} \in N^\infty$, we denote by $V_{g,n}(\kappa(\mathbf{b}))$ the higher Weil-Petersson volume

$$\langle \tau_0^n \kappa(\mathbf{b}) \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \kappa(\mathbf{b}).$$

Let $V_g(\kappa(\mathbf{b}))$ denote $V_{g,0}(\kappa(\mathbf{b}))$.

Higher Weil-Petersson volumes were extensively studied in the paper [13]. The authors found an explicit expression (see Lemma 2.2 below) of $V_{g,n}(\kappa(\mathbf{b}))$ in terms of integrals of ψ classes. In genus zero, they obtained more nice results about generating functions of $V_{0,n}(\kappa(\mathbf{b}))$ and raised the question whether their methods may be generalized to higher genera.

Although we feel it is difficult to generalize Kaufmann-Manin-Zagier's results to higher genera, we did find an effective recursion formula between $V_{g,n}(\kappa(\mathbf{b}))$ valid for all g and n, based on our previous work on integrals of ψ classes. The results are contained in the following two theorems.

Theorem 1.3. *Let $\mathbf{b} \in N^\infty$ and $n \geq 1$. Then*

$$\begin{aligned}(4)\quad \big(2g-1+||\mathbf{b}||\big)V_{g,n}(\kappa(\mathbf{b})) &= \frac{1}{12}V_{g-1,n+3}(\kappa(\mathbf{b})) - \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}} \binom{\mathbf{b}}{\mathbf{L}} V_{g,n}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}) \\ &+ \frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \mathbf{L}\neq\mathbf{0},\mathbf{L}'\neq\mathbf{0}}} \sum_{r+s=n-1} \binom{\mathbf{b}}{\mathbf{L}}\binom{n-1}{r} V_{g',r+2}(\kappa(\mathbf{L}))V_{g-g',s+2}(\kappa(\mathbf{L}')).\end{aligned}$$

Theorem 1.3 is an effective formula for computing higher Weil-Petersson volumes recursively by induction on g and $||\mathbf{b}||$, with the initial values

$$V_{0,3}(1) = 1 \quad \text{and} \quad V_{0,n}(\kappa(\boldsymbol{\delta}_{n-3})) = 1,\ n \geq 4,$$

where $\boldsymbol{\delta}_a$ denotes the sequence with 1 at the a-th place and zeros elsewhere.

Theorem 1.4. *Let $g \geq 2$ and $\mathbf{b} \in N^\infty$. Then*

$$\begin{aligned}(5)\quad \big((2g-1)(2g-2)+(4g-3)||\mathbf{b}||+||\mathbf{b}||^2\big)V_g(\kappa(\mathbf{b})) &= 5\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} V_{g,1}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|+1}) \\ -\frac{1}{6}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} V_{g-1,3}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}) &- \sum_{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}} V_{g',1}(\kappa_{|\mathbf{L}|}\kappa(\mathbf{e}))V_{g-g',2}(\kappa(\mathbf{f})) \\ &- (2g-1+||\mathbf{b}||)\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}} \binom{\mathbf{b}}{\mathbf{L}} V_g(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}) \\ &- \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}} \binom{\mathbf{b}}{\mathbf{L}} \sum_{\mathbf{e}+\mathbf{f}=\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|}} \binom{\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|}}{\mathbf{e}} V_g(\kappa(\mathbf{e})\kappa_{|\mathbf{f}|}).\end{aligned}$$

By induction on $||\mathbf{b}||$, Theorem 1.4 reduces the computation of $V_g(\kappa(\mathbf{b}))$ to the cases of $V_{g,n}(\kappa(\mathbf{b}))$ for $n \geq 1$, which have been computed by Theorem 1.3. Therefore Theorems 1.3 and 1.4 completely determine higher Weil-Petersson volumes of moduli spaces of curves.

The virtue of the above recursions is that they do not involve ψ classes. So if one wants to compute only higher Weil-Petersson volumes, the above recursions are more efficient both in speed and memory use, especially when we use "option remember" in a maple program.

On the other hand, we know that intersection numbers of mixed ψ and κ classes can be expressed by intersection numbers of pure κ classes [1].

In Section 2, we prove Theorems 1.1 and 1.2. In Section 3 we prove Theorems 1.3 and 1.4. In Section 4, we prove that the generating functions of intersection numbers involving general κ and ψ classes satisfy Virasoro constraints and the KdV hierarchy. In Section 5, we consider recursions of Hodge integrals with λ classes.

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2. Proofs of Theorems 1.1 and 1.2

The following elementary lemma is crucial to our proof.

Lemma 2.1. *Let $F(\mathbf{L}, n)$ and $G(\mathbf{L}, n)$ be two functions defined on $N^\infty \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. Let $\alpha_\mathbf{L}$ and $\beta_\mathbf{L}$ be real numbers depending only on $\mathbf{L} \in N^\infty$ that satisfy $\alpha_\mathbf{0}\beta_\mathbf{0} = 1$ and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_\mathbf{L}\beta_{\mathbf{L}'} = 0, \qquad \mathbf{b} \neq 0.$$

Then the following two identities are equivalent.

$$G(\mathbf{b}, n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_\mathbf{L} F(\mathbf{L}', n + |\mathbf{L}|), \quad \forall\ (\mathbf{b}, n) \in N^\infty \times \mathbb{N}$$
$$F(\mathbf{b}, n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \beta_\mathbf{L} G(\mathbf{L}', n + |\mathbf{L}|), \quad \forall\ (\mathbf{b}, n) \in N^\infty \times \mathbb{N}$$

Proof. Assume the first identity holds, we have

$$\begin{aligned}
\sum_{\mathbf{a}=\mathbf{0}}^{\mathbf{b}} \beta_\mathbf{a} G(\mathbf{b}-\mathbf{a}, n+|\mathbf{a}|) &= \sum_{\mathbf{a}=\mathbf{0}}^{\mathbf{b}} \beta_\mathbf{a} \sum_{\mathbf{a}'=\mathbf{0}}^{\mathbf{b}-\mathbf{a}} \alpha_{\mathbf{a}'} F(\mathbf{b}-\mathbf{a}-\mathbf{a}', n+|\mathbf{a}+\mathbf{a}'|) \\
&= \sum_{\mathbf{L}=\mathbf{0}}^{\mathbf{b}} \sum_{\mathbf{a}+\mathbf{a}'=\mathbf{L}} (\beta_\mathbf{a}\alpha_{\mathbf{a}'}) F(\mathbf{b}-\mathbf{L}, n+|\mathbf{L}|) \\
&= \sum_{\mathbf{L}=\mathbf{0}}^{\mathbf{b}} \delta_{\mathbf{L},\mathbf{0}} F(\mathbf{b}-\mathbf{L}, n+|\mathbf{L}|) \\
&= F(\mathbf{b}, n).
\end{aligned}$$

So we have proved the second identity. The proof of the other direction is the same. $\square$

We also need the following combinatorial formula from [13].

Lemma 2.2. [13] *Let $\mathbf{m} \in N^\infty$.*

$$\langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{m}) \rangle_g = \sum_{k=0}^{||\mathbf{m}||} \frac{(-1)^{||\mathbf{m}||-k}}{k!} \sum_{\substack{\mathbf{m}=\mathbf{m_1}+\cdots+\mathbf{m_k} \\ \mathbf{m_i}\neq\mathbf{0}}} \binom{\mathbf{m}}{\mathbf{m_1}, \dots, \mathbf{m_k}} \langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{|\mathbf{m_j}|+1} \rangle_g$$

$$= \sum_{k\geq 0} \sum_{\substack{\mathbf{m}=a_1\mathbf{m_1}+\cdots+a_k\mathbf{m_k}\\ \mathbf{m_i}\neq\mathbf{m_j}, i\neq j}} \frac{(-1)^{||\mathbf{m}||-\sum_{i=1}^k a_i}}{\prod_{i=1}^k a_i!} \binom{\mathbf{m}}{\underbrace{\mathbf{m_1},..,\mathbf{m_1}}_{a_1},\dots,\underbrace{\mathbf{m_k},..,\mathbf{m_k}}_{a_k}} \langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{|\mathbf{m_j}|+1}^{a_j} \rangle_g$$

where in the last term, these distinct $\{\mathbf{m_1},\dots,\mathbf{m_k}\}$ are unordered in the summation and a_i are positive integers.

Proof. We only give a sketch. Let $\pi_{n+p,n}:\overline{\mathcal{M}}_{g,n+p}\longrightarrow\overline{\mathcal{M}}_{g,n}$ be the morphism which forgets the last p marked points and denote $\pi_{n+p,n*}(\psi_{n+1}^{a_1+1}\dots\psi_{n+p}^{a_p+1})$ by $R(a_1,\dots,a_p)$, then we have the formula [1]

$$R(a_1,\dots,a_p) = \sum_{\sigma\in\mathbb{S}_p} \prod_{\substack{\text{each cycle } c\\ \text{of } \sigma}} \kappa_{\sum_{j\in c} a_j},$$

where we write any permutation σ in the symmetric group $\mathbb{S}_p$ as a product of disjoint cycles.

By a formal combinatorial argument, we get the following inversion result

$$\kappa_{a_1}\cdots\kappa_{a_p} = \sum_{k=1}^p \frac{(-1)^{p-k}}{k!} \sum_{\substack{\{1,\dots,p\}=S_1\coprod\cdots\coprod S_k\\ S_k\neq\emptyset}} R(\sum_{j\in S_1} a_j,\dots,\sum_{j\in S_k} a_j),$$

from which Lemma 2.2 follows. □

Proof of Theorem 1.1

Let LHS and RHS denote the left and right hand side of Theorem 1.1 respectively. By Lemma 2.2 and the Witten-Kontsevich theorem, we get

$$\begin{aligned}
&(2d_1+1)!!\langle\prod_{j=1}^n \tau_{d_j}\kappa(\mathbf{b})\rangle_g\\
&\quad= (2d_1+1)!!\sum_{k=0}^{||\mathbf{b}||}\frac{(-1)^{||\mathbf{b}||-k}}{k!}\sum_{\substack{\mathbf{m_1}+\cdots+\mathbf{m_k}=\mathbf{b}\\ \mathbf{m_i}\neq 0}}\binom{\mathbf{b}}{\mathbf{m_1},\dots,\mathbf{m_k}}\langle\prod_{j=1}^n\tau_{d_j}\prod_{j=1}^k\tau_{|\mathbf{m_j}|+1}\rangle_g\\
&\quad= \sum_{k=0}^{||\mathbf{b}||}\frac{(-1)^{||\mathbf{b}||-k}}{k!}\sum_{\substack{\mathbf{m_1}+\cdots+\mathbf{m_k}=\mathbf{b}\\ \mathbf{m_i}\neq 0}}\binom{\mathbf{b}}{\mathbf{m_1},\dots,\mathbf{m_k}}\\
&\qquad\times\Bigg(\sum_{j=2}^n\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\prod_{i=1}^k\tau_{|\mathbf{m_i}|+1}\rangle_g\\
&\qquad+\sum_{j=1}^k\frac{(2(d_1+|\mathbf{m_j}|)+1)!!}{(2|\mathbf{m_j}|+1)!!}\langle\tau_{d_1+|\mathbf{m_j}|}\prod_{i=2}^n\tau_{d_i}\prod_{i\neq j}\tau_{|\mathbf{m_i}|+1}\rangle_g\\
&\qquad+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\prod_{i=2}^n\tau_{d_i}\prod_{i=1}^k\tau_{|\mathbf{m_i}|+1}\rangle_{g-1}\\
&\qquad+\frac{1}{2}\sum_{\substack{I\coprod J=\{2,\dots,n\}\\ I'\coprod J'=\{1,\dots,k\}}}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\\
&\qquad\times\langle\tau_r\prod_{i\in I}\tau_{d_i}\prod_{i\in I'}\tau_{|\mathbf{m_i}|+1}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\prod_{i\in J'}\tau_{|\mathbf{m_i}|+1}\rangle_{g-g'}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=2}^{n} \frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!} \langle \kappa(\mathbf{b}) \tau_{d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i} \rangle_g \\
&\qquad + \frac{1}{2} \sum_{r+s=|d_1|-2} (2r+1)!!(2s+1)!! \langle \kappa(\mathbf{b}) \tau_r \tau_s \prod_{i\neq 1} \tau_{d_i} \rangle_{g-1} \\
&\qquad + \frac{1}{2} \sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I \coprod J=\{2,\dots,n\}}} \sum_{r+s=d_1-2} \binom{\mathbf{b}}{\mathbf{e}} (2r+1)!!(2s+1)!! \\
&\qquad\qquad \times \langle \kappa(\mathbf{e}) \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f}) \tau_s \prod_{i\in J} \tau_{d_i} \rangle_{g-g'} \\
&\qquad + \sum_{k=0}^{||\mathbf{b}||} \frac{(-1)^{||\mathbf{b}||-k}}{k!} \sum_{\substack{\mathbf{m_1}+\cdots+\mathbf{m_k}=\mathbf{b} \\ \mathbf{m_i}\neq 0}} \binom{\mathbf{b}}{\mathbf{m_1},\dots,\mathbf{m_k}} \\
&\qquad \times \sum_{j=1}^{k} \frac{(2(d_1+|\mathbf{m_j}|)+1)!!}{(2|\mathbf{m_j}|+1)!!} \langle \tau_{d_1+|\mathbf{m_j}|} \prod_{i=2}^{n} \tau_{d_i} \prod_{i\neq j} \tau_{|\mathbf{m_i}|+1} \rangle_g \\
&= RHS + \sum_{k\geq 0} \frac{(-1)^{||\mathbf{b}||-k-1}}{(k+1)!} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}\neq \mathbf{0}}} \sum_{\substack{\mathbf{m_1}+\cdots+\mathbf{m_k}=\mathbf{b}-\mathbf{L} \\ \mathbf{m_i}\neq 0}} \binom{\mathbf{b}}{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{m_1},\dots,\mathbf{m_k}} \\
&\qquad \times (k+1) \frac{(2(d_1+|\mathbf{L}|)+1)!!}{(2|\mathbf{L}|+1)!!} \langle \tau_{d_1+|\mathbf{L}|} \prod_{i=2}^{n} \tau_{d_i} \prod_{i=1}^{k} \tau_{|\mathbf{m_i}|+1} \rangle_g \\
&= RHS - \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}\neq \mathbf{0}}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d_1+2|\mathbf{L}|+1)!!}{(2|\mathbf{L}|+1)!!} \langle \kappa(\mathbf{L}') \tau_{d_1+|\mathbf{L}|} \prod_{j=2}^{n} \tau_{d_j} \rangle_g \\
&\qquad\qquad = RHS - LHS + (2d_1+1)!! \langle \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{b}) \rangle_g.
\end{aligned}
$$

In the third equation, only the quadratic term needs a careful verification. So we have proved $RHS = LHS$.

We will see that Theorem 1.2 follows from Theorem 1.1 and Lemma 2.1.

Proof of Theorem 1.2

Let

$$F(\mathbf{b}, d_1) = \frac{(2d_1+1)!!}{\mathbf{b}!} \langle \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{b}) \rangle_g$$

and

$$
\begin{aligned}
G(\mathbf{b}, d_1) &= \sum_{j=2}^{n} \frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{\mathbf{b}!(2d_j-1)!!} \langle \kappa(\mathbf{b}) \tau_{d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i} \rangle_g \\
&\qquad + \frac{1}{2} \sum_{r+s=d_1-2} \frac{(2r+1)!!(2s+1)!!}{\mathbf{b}!} \langle \kappa(\mathbf{b}) \tau_r \tau_s \prod_{i=2}^{n} \tau_{d_i} \rangle_{g-1}
\end{aligned}
$$

$$+\frac{1}{2}\sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b}\\ I\coprod J=\{2,\dots,n\}}}\sum_{r+s=d_1-2}\frac{(2r+1)!!(2s+1)!!}{\mathbf{e}!\mathbf{f}!}$$

$$\times\langle\kappa(\mathbf{e})\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa(\mathbf{f})\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

Note that Theorem 1.1 is just

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!}F(\mathbf{L}',d_1+|\mathbf{L}|)=G(\mathbf{b},d_1).$$

By Lemma 2.1, we have

$$F(\mathbf{b},d_1)=\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\frac{\alpha_{\mathbf{L}}}{\mathbf{L}!}G(\mathbf{L}',d_1+|\mathbf{L}|),$$

which is just the result we want.

3. Higher Weil-Petersson volumes

By applying Lemma 2.2 as in the proof of Theorem 1.1, we may generalize recursions of pure ψ classes to recursions including both ψ and κ classes.

First we have the following generalization of the string and dilation equations.

Proposition 3.1. *For* $\mathbf{b}\in N^\infty$, $n\geq 0$ *and* $d_j\geq 0$,

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|}\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{L}')\rangle_g=\sum_{j=1}^n\langle\tau_{d_j-1}\prod_{i\neq j}\tau_{d_i}\kappa(\mathbf{b})\rangle_g,$$

and

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|+1}\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{L}')\rangle_g=(2g-2+n)\langle\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{b})\rangle_g.$$

Proof. The first identity follows by taking $d_1=0$ in Theorem 1.1. For the second identity, we have

$$\langle\prod_{j=1}^n\tau_{d_j}\tau_1\kappa(\mathbf{b})\rangle_g$$

$$=\sum_{k\geq 0}\sum_{\substack{\mathbf{m_1}+\dots+\mathbf{m_k}=\mathbf{b}\\ \mathbf{m_i}\neq 0}}\frac{(-1)^{||\mathbf{b}||-k}}{k!}\binom{\mathbf{b}}{\mathbf{m_1}\dots,\mathbf{m_k}}\langle\tau_1\prod_{j=1}^n\tau_{d_j}\prod_{j=1}^k\tau_{|\mathbf{m}_j|+1}\rangle_g$$

$$=(2g+n-2)\langle\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{b})\rangle_g$$

$$+\sum_{k\geq 0}\sum_{\substack{\mathbf{L}+\mathbf{m_1}+\dots+\mathbf{m_k}=\mathbf{b}\\ \mathbf{L}\neq 0,\mathbf{m_i}\neq 0}}\frac{(-1)^{||\mathbf{b}||-k-1}}{k!}\binom{\mathbf{b}}{\mathbf{L},\mathbf{m_1}\dots,\mathbf{m_k}}\langle\tau_{|\mathbf{L}|+1}\prod_{j=1}^k\tau_{|\mathbf{m}_j|+1}\prod_{j=1}^n\tau_{d_j}\rangle_g.$$

Subtracting the last term from each side, we have proved the second identity. □

For the particular case $\mathbf{b}=(m,0,0,\dots)$, Proposition 3.1 has been proved by Norman Do and Norbury [3] in their study of the intermediary moduli spaces consisting of hyperbolic surfaces with a cone point of a specified angle.

We need the following results from [1].

Lemma 3.2. *Let $\pi_{n+1}:\overline{\mathcal{M}}_{g,n+1}\longrightarrow\overline{\mathcal{M}}_{g,n}$ be the morphism that forgets the last marked point.*

i) $\pi_{n*}(\psi_1^{a_1}\cdots\psi_{n-1}^{a_{n-1}}\psi_n^{a_n+1})=\psi_1^{a_1}\cdots\psi_{n-1}^{a_{n-1}}\kappa_{a_n}\qquad$ *for* $\quad a_j\geq 0$;

ii) $\kappa_a=\pi_{n+1}^*(\kappa_a)+\psi_{n+1}^a\qquad$ *on* $\quad\overline{\mathcal{M}}_{g,n+1}$;

iii) $\kappa_0=2g-2+n\qquad$ *on* $\quad\overline{\mathcal{M}}_{g,n}$.

We have the following generalization of a recursion formula from the Witten-Kontsevich theorem corresponding to the first equation in the KdV hierarchy (see Theorem 1.2 of [15]).

Proposition 3.3. *Let $\mathbf{b}\in N^\infty$ and $n\geq 0$. Then*

$$\begin{aligned}(6)\quad \langle\tau_0\tau_1\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{b})\rangle_g=&\frac{1}{12}\langle\tau_0^4\prod_{j=1}^n\tau_{d_j}\kappa(\mathbf{b})\rangle_{g-1}\\&+\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \underline{n}=I\coprod J}}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_0^2\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\rangle_{g-g'}.\end{aligned}$$

Now we give a proof of Theorem 1.3. Let LHS and RHS denote the left and right hand side of Proposition 3.3 respectively. Taking $d_j=0$ and applying Lemma 3.2, we have

$$\begin{aligned}LHS=&\int_{\overline{\mathcal{M}}_{g,n+1}}\pi_{n+2*}\left(\psi_{n+2}\prod_{i\geq1}(\pi_{n+2}^*\kappa_i+\psi_{n+2}^i)^{b(i)}\right)\\&=\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_0^{n+1}\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}\rangle_g\\&=\big((2g-1+n)+||\mathbf{b}||\big)\langle\tau_0^{n+1}\kappa(\mathbf{b})\rangle_g+\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq2}}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_0^{n+1}\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}\rangle_g\end{aligned}$$

and

$$\begin{aligned}RHS=&\frac{1}{12}\langle\tau_0^{n+4}\kappa(\mathbf{b})\rangle_{g-1}+\frac{1}{2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\sum_{r+s=n}\binom{\mathbf{b}}{\mathbf{L}}\binom{n}{r}\langle\tau_0^{r+2}\kappa(\mathbf{L})\rangle_{g'}\langle\tau_0^{s+2}\kappa(\mathbf{L}')\rangle_{g-g'}\\=&\frac{1}{12}\langle\tau_0^{n+4}\kappa(\mathbf{b})\rangle_{g-1}+\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \mathbf{L}\neq\mathbf{0},\mathbf{L}'\neq\mathbf{0}}}\sum_{r+s=n}\binom{\mathbf{b}}{\mathbf{L}}\binom{n}{r}\langle\tau_0^{r+2}\kappa(\mathbf{L})\rangle_{g'}\langle\tau_0^{s+2}\kappa(\mathbf{L}')\rangle_{g-g'}\\&+n\langle\tau_0^{n+1}\kappa(\mathbf{b})\rangle_g.\end{aligned}$$

So Theorem 1.3 follows from $LHS=RHS$.

By further expanding the term $V_{g-1,n+3}(\kappa(\mathbf{b}))$ in Theorem 1.3, we get

$$\begin{aligned}V_{g,n}(\kappa(\mathbf{b}))=&\delta_{||\mathbf{b}||,0}+\frac{1}{24^g g!}\delta_{||\mathbf{b}||,1}+\sum_{h=0}^g\frac{(2h-3+||\mathbf{b}||)!!}{12^{g-h}(2g-1+||\mathbf{b}||)!!}\times\\&\Bigg(\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \mathbf{L}\neq\mathbf{0},\mathbf{L}'\neq\mathbf{0}}}\sum_{r+s=n-1+3(g-h)}\binom{\mathbf{b}}{\mathbf{L}}\binom{n-1+3(g-h)}{r}V_{h',r+2}(\kappa(\mathbf{L}))V_{h-h',s+2}(\kappa(\mathbf{L}'))\end{aligned}$$

$$-\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}}\binom{\mathbf{b}}{\mathbf{L}}V_{h,n+3(g-h)}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|})\Bigg).$$

The following proposition is a generalization of a recursion formula proved in Proposition 2.6 of [15].

Proposition 3.4. *Let $\mathbf{b}\in N^\infty$, $n\geq 0$ and $r\geq 0$. Then*

$$\begin{aligned}(7)\quad \langle\tau_1\tau_r\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{b})\rangle_g &= (2r+3)\langle\tau_0\tau_{r+1}\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{b})\rangle_g-\frac{1}{6}\langle\tau_0^3\tau_r\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{b})\rangle_{g-1}\\ &\quad -\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \underline{n}=I\coprod J}}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_0\tau_r\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\rangle_{g-g'}.\end{aligned}$$

Let LHS and RHS denote the left and right hand side of Proposition 3.4 respectively. Taking $r=1$ and $n=0$, we have

$$\begin{aligned}LHS &= \int_{\overline{\mathcal{M}}_{g,1}}\pi_{2*}\left(\psi_1\psi_2\prod_{i\geq 1}(\pi_2^*\kappa_i+\psi_2^i)^{b(i)}\right)\\ &= \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\binom{\mathbf{b}}{\mathbf{L}}\int_{\overline{\mathcal{M}}_{g,1}}\psi_1\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}\\ &= (||\mathbf{b}||+2g-1)\int_{\overline{\mathcal{M}}_{g,1}}\psi_1\kappa(\mathbf{b})+\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}}\int_{\overline{\mathcal{M}}_{g,1}}\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|}\\ &= \big((2g-1)(2g-2)+(4g-3)||\mathbf{b}||+||\mathbf{b}||^2\big)V_g(\kappa(\mathbf{b}))\\ &\quad +(2g-1+||\mathbf{b}||)\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}}\binom{\mathbf{b}}{\mathbf{L}}V_g(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|})\\ &\quad +\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ ||\mathbf{L}'||\geq 2}}\binom{\mathbf{b}}{\mathbf{L}}\sum_{\mathbf{e}+\mathbf{f}=\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|}}\binom{\mathbf{L}+\boldsymbol{\delta}_{|\mathbf{L}'|}}{\mathbf{e}}V_g(\kappa(\mathbf{e})\kappa_{|\mathbf{f}|}).\end{aligned}$$

and similarly,

$$\begin{aligned}RHS &= 5\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\binom{\mathbf{b}}{\mathbf{L}}V_{g,1}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|+1})-\frac{1}{6}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\binom{\mathbf{b}}{\mathbf{L}}V_{g-1,3}(\kappa(\mathbf{L})\kappa_{|\mathbf{L}'|})\\ &\quad -\sum_{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}}\binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}}V_{g',1}(\kappa_{|\mathbf{L}|}\kappa(\mathbf{e}))V_{g-g',2}(\kappa(\mathbf{f})).\end{aligned}$$

So we have proved Theorem 1.4.

4. Virasoro constraints and the KdV hierarchy

In this section, we follow the arguments of Mulase and Safnuk [20] to study properties of generating functions of intersections of ψ and κ classes using Theorems 1.1 and 1.2.

Let $\mathbf{s} := (s_1, s_2, \dots)$ and $\mathbf{t} := (t_0, t_1, t_2, \dots)$, we introduce the following generating function

$$G(\mathbf{s}, \mathbf{t}) := \sum_g \sum_{\mathbf{m},\mathbf{n}} \langle \kappa_1^{m_1} \kappa_2^{m_2} \cdots \tau_0^{n_0} \tau_1^{n_1} \cdots \rangle_g \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!},$$

where $\mathbf{s}^{\mathbf{m}} = \prod_{i \geq 1} s_i^{m_i}$.

Propositions 3.3 and 3.4 can be reformulated in terms of differential operators.

Proposition 4.1. *Let $r \geq 0$. Then we have*

$$\frac{\partial^2 G}{\partial t_0 \partial t_1} = \frac{1}{12} \frac{\partial^4 G}{\partial t_0^4} + \frac{1}{2} \frac{\partial^2 G}{\partial t_0^2} \frac{\partial^2 G}{\partial t_0^2}$$

and

$$\frac{\partial^2 G}{\partial t_1 \partial t_r} = (2r+3) \frac{\partial^2 G}{\partial t_0 \partial t_{r+1}} - \frac{1}{6} \frac{\partial^4 G}{\partial t_0^3 \partial t_r} - \frac{\partial^2 G}{\partial t_0 \partial t_r} \frac{\partial^2 G}{\partial t_0^2}.$$

We define $\beta_{\mathbf{L}} = \alpha_{\mathbf{L}}/\mathbf{L}!$ where $\alpha_{\mathbf{L}}$ are the same constants in Theorem 1.2. We introduce the following family of differential operators for $k \geq -1$,

$$\begin{aligned}(8)\quad \hat{V}_k = &-\frac{(2k+3)!!}{2} \frac{\partial}{\partial t_{k+1}} + \delta_{k,-1}\left(\frac{t_0^2}{4} + \frac{s_1}{48}\right) + \frac{\delta_{k,0}}{16} \\ &+ \frac{1}{2} \sum_{\mathbf{L}} \sum_{j=0}^{\infty} \frac{(2(|\mathbf{L}|+j+k)+1)!!}{(2j-1)!!} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} t_j \frac{\partial}{\partial t_{|\mathbf{L}|+j+k}} \\ &+ \frac{1}{4} \sum_{\mathbf{L}} \sum_{\substack{d_1+d_2=\\|\mathbf{L}|+k-1}} (2d_1+1)!!(2d_2+1)!! \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}}.\end{aligned}$$

Theorem 4.2. *We have $\hat{V}_k \exp(G) = 0$ for $k \geq -1$ and*

$$[\hat{V}_n, \hat{V}_m] = (n-m) \sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \hat{V}_{n+m+|\mathbf{L}|}.$$

Proof. Note that the termination cases of the recursion formula in Theorem 1.2 are

$$\langle \tau_0 \kappa_1 \rangle_1 = \frac{1}{24}, \qquad \langle \tau_0^3 \rangle_0 = 1, \qquad \langle \tau_1 \rangle_1 = \frac{1}{24}.$$

So $\hat{V}_k \exp(G) = 0$ for $k \geq -1$ is just a restatement of Theorem 1.2.

One may check directly that

$$[\hat{V}_n, \hat{V}_m] = (n-m) \sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \hat{V}_{n+m+|\mathbf{L}|}.$$

□

The following constants are inverse to $\beta_{\mathbf{L}}$,

$$\gamma_{\mathbf{L}} := \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!}.$$

Define a new family of differential operators V_k for $k \geq -1$ by

$$\begin{aligned}(9)\quad V_k = &-\frac{1}{2} \sum_{\mathbf{L}} (2(|\mathbf{L}|+k)+3)!! \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \frac{\partial}{\partial t_{|\mathbf{L}|+k+1}} + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}} \\ &+ \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1+1)!!(2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}} + \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{16},\end{aligned}$$

Theorem 1.1 implies $V_k\exp(G)=0$. We now prove that the operators V_k satisfy the Virasoro relations

$$[V_n,V_m]=(n-m)V_{n+m}.$$

Introduce new variables

$$T_{2i+1}:=\frac{t_i}{(2i+1)!!},\quad i\geq 0$$

which transform the operators $\hat{V}_k$ into

$$\begin{aligned}\hat{V}_k=&-\frac{1}{2}\frac{\partial}{\partial T_{2k+3}}+\delta_{k,-1}(\frac{t_0^2}{4}+\frac{s_1}{48})+\frac{\delta_{k,0}}{16}\\&+\frac{1}{2}\sum_{\mathbf{L}}\sum_{j=0}^{\infty}(2j+1)\beta_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}T_{2j+1}\frac{\partial}{\partial T_{2(|\mathbf{L}|+j+k)+1}}\\&+\frac{1}{4}\sum_{\mathbf{L}}\sum_{\substack{d_1+d_2=\\|\mathbf{L}|+k-1}}\beta_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}\frac{\partial^2}{\partial T_{2d_1+1}\partial T_{2d_2+1}}.\end{aligned}$$

Define operators J_p for $p\in\mathbb{Z}$ by

$$J_p=\begin{cases}(-p)T_{-p} & \text{if } p<0,\\ \frac{\partial}{\partial T_p} & \text{if } p>0.\end{cases}$$

Then

$$\hat{V}_k=-\frac{1}{2}J_{2k+3}+\sum_{\mathbf{L}}\beta_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}E_{k+|\mathbf{L}|},$$

where

$$E_k=\frac{1}{4}\sum_{p\in\mathbb{Z}}J_{2p+1}J_{2(k-p)-1}+\frac{\delta_{k,0}}{16}.$$

It's not difficult to see that

$$V_k=\sum_{\mathbf{L}}\gamma_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}\hat{V}_{k+|\mathbf{L}|}=-\frac{1}{2}\sum_{\mathbf{L}}\gamma_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}J_{2k+2|\mathbf{L}|+3}+E_k.$$

Theorem 4.3. *The operators V_k, $k\geq -1$ satisfy the Virasoro relations*

$$[V_n,V_m]=(n-m)V_{n+m}.$$

Proof. Since

$$\begin{aligned}E_k=&\frac{1}{2}\sum_{j=0}^{\infty}\frac{(2(j+k)+1)!!}{(2j-1)!!}t_j\frac{\partial}{\partial t_{j+k}}\\&+\frac{1}{4}\sum_{d_1+d_2=k-1}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial t_{d_1}\partial t_{d_2}}+\frac{\delta_{k,-1}t_0^2}{4}+\frac{\delta_{k,0}}{16}.\end{aligned}$$

We can check directly that

$$[E_n,E_m]=(n-m)E_{n+m},\qquad [J_{2k+3},E_m]=\frac{2k+3}{2}J_{2(k+m)+3}.$$

So we have

$$[V_n,V_m]=\big[-\frac{1}{2}\sum_{\mathbf{L}}\gamma_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}J_{2(n+|\mathbf{L}|)+3}+E_n,-\frac{1}{2}\sum_{\mathbf{L}}\gamma_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}J_{2(m+|\mathbf{L}|)+3}+E_m\big]$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{\mathbf{L}} \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \left([J_{2(n+|\mathbf{L}|)+3}, E_m] + [E_n, J_{2(m+|\mathbf{L}|)+3}] \right) + [E_n, E_m] \\
&= -\frac{1}{2}\sum_{\mathbf{L}} \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} (n-m) J_{2(n+m+|\mathbf{L}|)+3} + (n-m)E_{n+m} \\
&= (n-m)V_{n+m}.
\end{aligned}
$$

□

Now we recall the KdV hierarchy, which is the following hierarchy of differential equations for $n \geq 1$,

$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1},$$

where R_n are polynomials in $U, \partial U/\partial t_0, \partial^2 U/\partial t_0^2, \dots$, which is defined recursively by

$$R_1 = U, \qquad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1}\left(\frac{\partial U}{\partial t_0} R_n + 2U \frac{\partial R_n}{\partial t_0} + \frac{1}{4}\frac{\partial^3}{\partial t_0^3} R_n \right).$$

In particular, it is easy to see that

$$R_2 = \frac{1}{2}U^2 + \frac{1}{12}\frac{\partial^2 U}{\partial t_0^2},$$

so the first equation in the KdV hierarchy is the classical KdV equation

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12}\frac{\partial^3 U}{\partial t_0^3}.$$

The Witten-Kontsevich theorem [25, 14] states that the generating function for ψ class intersections

$$F(t_0, t_1, \dots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ-function for the KdV hierarchy, i.e. $\partial^2 F/\partial t_0^2$ obeys all equations in the KdV hierarchy.

Theorem 4.4. *We have*

$$G(\mathbf{s}, t_0, t_1, \dots) = F(t_0, t_1, t_2 + p_2, t_3 + p_3, \dots), \tag{10}$$

where p_k are polynomials in $\mathbf{s}$ given by

$$p_k = -\sum_{|\mathbf{L}|=k-1} (2|\mathbf{L}|+1)!! \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{||\mathbf{L}||-1}}{\mathbf{L}!} \mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of $\mathbf{s}$, $G(\mathbf{s}, \mathbf{t})$ is a τ-function for the KdV hierarchy.

Proof. The change of variables

$$\tilde{t}_i = \begin{cases} t_i & \text{for } i = 0, 1 \ , \\ t_i - \sum_{|\mathbf{L}|=i-1} (2|\mathbf{L}|+1)!! \gamma_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} & \text{otherwise,} \end{cases}$$

transforms the operators V_k of (9) into

$$
\begin{aligned}
V_k = &-\frac{1}{2}(2k+3)!! \frac{\partial}{\partial \tilde{t}_{k+1}} + \frac{1}{2}\sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} \tilde{t}_j \frac{\partial}{\partial \tilde{t}_{j+k}} \\
&+ \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1+1)!!(2d_2+1)!! \frac{\partial^2}{\partial \tilde{t}_{d_1} \partial \tilde{t}_{d_2}} + \frac{\delta_{k,-1} \tilde{t}_0^2}{4} + \frac{\delta_{k,0}}{16},
\end{aligned}
$$

which is just the operator obtained by setting $\mathbf{s} = \mathbf{0}$ in $\hat{V}_k$ of (8). Since Virasoro constraints uniquely determine the generating functions $G(\mathbf{s}, t_0, t_1, \dots)$ and $F(t_0, t_1, \dots)$, we have for any fixed values of $\mathbf{s}$,

$$G(\mathbf{s}, t_0, t_1, t_2, \dots) = F(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2, \dots).$$

So we have proved the theorem. $\square$

Theorem 4.4 can also be proved directly by applying Lemma 2.2, as discussed in [17].

5. Tautological constants of Hodge integrals

The results in this section can be applied to study Faber's perfect pairing conjecture [8] and its generalizations.

Let $\mathcal{M}_{g,n}^{rt}$ be the moduli space of "curves with rational tails"(i.e. with dual graph with a vertex of genus g). Let $\mathcal{M}_{g,n}^{ct}$ be the moduli space of "curves of compact type", (i.e. with dual graph with no loops). Hence

$$\mathcal{M}_{g,n}^{rt} \subset \mathcal{M}_{g,n}^{ct} \subset \overline{\mathcal{M}}_{g,n}.$$

Conjecture 5.1. *(Faber, Hain, Looijenga, Pandharipande, et al.) The space $\overline{\mathcal{M}}_{g,n}$ (resp. $\mathcal{M}_{g,n}^{rt}$, $\mathcal{M}_{g,n}^{ct}$) "behaves like" a complex variety of dimension $D = 3g-3+n$ (resp. $g-2+n$, $2g-3+n$). More precisely, its tautological ring R^* has the following properties.*

- Socle statement: *$R^i = 0$ for $i > D$, $R^D \cong \mathbb{Q}$, and*
- Perfect pairing statement: *for $0 \le i \le D$, the natural map $R^i \times R^{D-i} \to R^D$ is a perfect pairing.*

The socle statement has been proved by Graber and Vakil [11]. While the perfect paring statement is still open.

By the above conjecture, tautological relations in $\mathcal{M}_{g,n}^{rt}$ and $\mathcal{M}_{g,n}^{ct}$ are determined respectively by the following linear functionals, called intersection pairings.

$$\begin{aligned} R^i(\mathcal{M}_{g,n}^{rt}) \times R^{g-2+n-i}(\mathcal{M}_{g,n}^{rt}) &\longrightarrow \mathbb{Q} \\ (u,v) &\longmapsto \int_{\overline{\mathcal{M}}_{g,n}} uv\lambda_g\lambda_{g-1}, \end{aligned}$$

and

$$\begin{aligned} R^i(\mathcal{M}_{g,n}^{ct}) \times R^{2g-3+n-i}(\mathcal{M}_{g,n}^{ct}) &\longrightarrow \mathbb{Q} \\ (u,v) &\longmapsto \int_{\overline{\mathcal{M}}_{g,n}} uv\lambda_g. \end{aligned}$$

Since tautological classes are represented by linear combinations of decorated stable graphs, the computation of intersection pairings will eventually reduce to the following integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{b_1}\cdots\kappa_{b_k}\psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_g\lambda_{g-1},$$

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{b_1}\cdots\kappa_{b_k}\psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_g.$$

Commonly, one would compute the above integrals by first eliminating κ classes, then applying the $\lambda_g\lambda_{g-1}$ theorem or the λ_g theorem.

Now we present more efficient recursion formulae computing these integrals, their patterns may well give some implications of the perfect pairing conjectures.

From degree 0 Virasoro constraints for a surface, Getzler and Pandharipande [9] obtained the following recursion.

Lemma 5.2. [9] *Let $d, d_0 \geq 0$ and $d_j \geq 1$ for $j \geq 1$.*

$$\langle \tau_d \tau_{d_0} \prod_{j=1}^{n} \tau_{d_j} \mid \lambda_g \lambda_{g-1} \rangle_g = \frac{(2d+2d_0-1)!!}{(2d-1)!!(2d_0-1)!!} \langle \tau_{d_0+d-1} \prod_{j=1}^{n} \tau_{d_j} \mid \lambda_g \lambda_{g-1} \rangle_g$$
$$+ \sum_{j=1}^{n} \frac{(2d+2d_j-3)!!}{(2d-1)!!(2d_j-3)!!} \langle \tau_{d_0} \tau_{d_j+d-1} \prod_{i\neq j} \tau_{d_i} \mid \lambda_g \lambda_{g-1} \rangle_g$$

Lemma 5.2 has the following generalization.

Theorem 5.3. *Let $\mathbf{b} \in N^\infty$, $d, d_0 \geq 0$ and $d_j \geq 1$ for $j \geq 1$. Then*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d+2|\mathbf{L}|-1)!!}{(2|\mathbf{L}|-1)!!} \langle \tau_{d+|\mathbf{L}|} \tau_{d_0} \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g$$
$$= \frac{(2d+2d_0-1)!!}{(2d_0-1)!!} \langle \tau_{d_0+d-1} \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{b}) \mid \lambda_g \lambda_{g-1} \rangle_g$$
$$+ \sum_{j=1}^{n} \frac{(2d+2d_j-3)!!}{(2d_j-3)!!} \langle \tau_{d_0} \tau_{d_j+d-1} \prod_{i\neq j} \tau_{d_i} \kappa(\mathbf{b}) \mid \lambda_g \lambda_{g-1} \rangle_g$$

and

$$\langle \tau_d \tau_{d_0} \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{b}) \mid \lambda_g \lambda_{g-1} \rangle_g$$
$$= \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \gamma_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d+2d_0+2|\mathbf{L}|-1)!!}{(2d-1)!!(2d_0-1)!!} \langle \tau_{d_0+d+|\mathbf{L}|-1} \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g$$
$$+ \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{j=1}^{n} \gamma_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d+2d_j+2|\mathbf{L}|-3)!!}{(2d-1)!!(2d_j-3)!!} \langle \tau_{d_0} \tau_{d_j+d+|\mathbf{L}|-1} \prod_{i\neq j} \tau_{d_i} \kappa(\mathbf{L}') \mid \lambda_g \lambda_{g-1} \rangle_g$$

where $\gamma_{\mathbf{L}} \in \mathbb{Q}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{||\mathbf{L}||} \gamma_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|-1)!!} = 0, \qquad \mathbf{b} \neq 0,$$

with the initial value $\gamma_{\mathbf{0}} = 1$.

Corollary 5.4. *In Theorem 5.3, we have*

$$\gamma_l = \frac{E_l}{(2l-1)!!}, \quad \gamma(\underbrace{0, \ldots, 0, 1}_{l}) = \frac{1}{(2l-1)!!}$$

where E_l are the Euler numbers that satisfy

$$\sec x = \frac{1}{\cos x} = \sum_{k=0}^{\infty} \frac{E_k}{(2k)!} x^{2k} = 1 + \frac{1}{2!}x^2 + \frac{5}{4!}x^4 + \frac{61}{6!}x^6 + \frac{1385}{8!}x^8 + \frac{50521}{10!}x^{10} + \cdots.$$

Proof. We have

$$\cos(\sqrt{2}x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k-1)!!} x^{2k},$$

by Theorem 5.3,

$$\sec(\sqrt{2}x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^{2k}.$$

So we get the formula of γ_l. □

The following recursion follows from degree 0 Virasoro constraints for a curve.

Lemma 5.5. [9] *Let $d, d_0 \geq 0$ and $d_j \geq 1$ for $j \geq 1$.*

$$\langle \tau_d \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \mid \lambda_g \rangle_g = \binom{d+d_0}{d_0} \langle \tau_{d_0+d-1} \prod_{j=1}^n \tau_{d_j} \mid \lambda_g \rangle_g + \sum_{j=1}^n \binom{d_j+d-1}{d_j-1} \langle \tau_{d_0} \tau_{d_j+d-1} \prod_{i \neq j} \tau_{d_i} \mid \lambda_g \rangle_g,$$

Lemma 5.5 has the following generalization.

Theorem 5.6. *Let $\mathbf{b} \in N^\infty$, $d, d_0 \geq 0$ and $d_j \geq 1$ for $j \geq 1$.*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} (-1)^{||\mathbf{L}||} \frac{(d+|\mathbf{L}|)!}{|\mathbf{L}|!} \langle \tau_d \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \rangle_g = \frac{(d+d_0)!}{d_0!} \langle \tau_{d_0+d-1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \mid \lambda_g \rangle_g + \sum_{j=1}^n \frac{(d_j+d-1)!}{(d_j-1)!} \langle \tau_{d_0} \tau_{d_j+d-1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{b}) \mid \lambda_g \rangle_g$$

and

$$\langle \tau_d \tau_{d_0} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \mid \lambda_g \rangle_g = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \gamma_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(d+d_0+|\mathbf{L}|)!}{d_0! d!} \langle \tau_{d_0+d+|\mathbf{L}|-1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \mid \lambda_g \rangle_g + \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{j=1}^n \gamma_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(d_j+d+|\mathbf{L}|-1)!}{(d_j-1)! d!} \langle \tau_{d_0} \tau_{d_j+d+|\mathbf{L}|-1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{L}') \mid \lambda_g \rangle_g$$

where $\gamma_{\mathbf{L}} \in \mathbb{Q}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{||\mathbf{L}||} \gamma_{\mathbf{L}}}{\mathbf{L}! \mathbf{L}'! |\mathbf{L}'|!} = 0, \qquad \mathbf{b} \neq 0,$$

with the initial value $\gamma_{\mathbf{0}} = 1$.

We recall the definition of the Bessel functions of the first kind. For the Bessel equations of order ν

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0,$$

we have the following solutions

$$y = J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}.$$

These are called Bessel functions of the first kind of order ν.

Corollary 5.7. *In Theorem 5.6, we have*

$$\gamma(\underbrace{0, \ldots, 0, 1}_{l}) = \frac{1}{l!}$$

and γ_l is given by

$$\frac{1}{J_0(\sqrt{4x})} = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = 1 + x + \frac{3/2}{2!}x^2 + \frac{19/6}{3!}x^3 + \frac{211/24}{4!}x^4 + \frac{1217/40}{5!}x^5 + \cdots,$$

where J_0 is the Bessel function of the first kind of order zero

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2} x^{2k}.$$

Proof. The corollary follows easily from Theorem 5.6 and the following

$$J_0(\sqrt{4x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} x^k.$$

□

It is interesting to notice that the Bessel function of the first kind of order zero also appears in Manin and Zograf's work [17] on asymptotics for Weil-Petersson volumes.

References

[1] E. Arbarello and M. Cornalba, *Combinatorial and Algebro-Geometric cohomology classes on the Moduli Spaces of Curves*, J. Algebraic Geometry, **5** (1996), 705–709.

[2] E. Arbarello and M. Cornalba, *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Publications Mathématiques de l'IHÉS, 88 (1998), 97–127.

[3] Norman Do and P. Norbury, *Weil-Petersson volumes and cone surfaces*, to appear in Geometriae Dedicata.

[4] R. Dijkgraaf, H. Verlinde, and E. Verlinde, *Topological strings in $d < 1$*, Nuclear Phys. B 352 (1991), 59–86.

[5] B. Eynard and N. Orantin, *Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models*, math-ph/0705.3600.

[6] B. Eynard, *Recursion between Mumford volumes of moduli spaces*, math-ph/0706.4403.

[7] C. Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians*, in New Trends in Algebraic Geometry (K. Hulek, F. Catanese, C. Peters and M. Reid, eds.), 93–109, Cambridge University Press, 1999.

[8] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves.* In Moduli of curves and abelian varieties, Aspects Math., E33, Vieweg, Braunschweig, Germany, 1999. 109–129.

[9] E. Getzler, R. Pandharipande, *Virasoro constraints and the Chern classes of the Hodge bundle*, Nuclear Phys. B **530** (1998), no. 3, 701–714.

[10] S. Grushevsky, *An explicit upper bound for Weil-Petersson volumes of the moduli spaces of punctured Riemann surfaces*, Math. Ann. 321 (2001), 1–13.

[11] T. Graber and R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. 30 (2005), 1–37.

[12] A. Kabanov and T. Kimura, *Intersection Numbers and Rank One Cohomological Field Theories in Genus One*, Commun. Math. Phys. 194 (1998), 651-674.

[13] R. Kaufmann, Yu. Manin, and D. Zagier, *Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves*, Comm. Math. Phys. 181 (1996), 763-787.

[14] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function.* Comm. Math. Phys. **147** (1992), no. 1, 1–23.

[15] K. Liu and H. Xu, *An effective recursion formula for computing intersection numbers*, math.AG/0710.5322.

[16] K. Liu and H. Xu, *Mirzakhani's recursion formula is equivalent to the Witten-Kontsevich theorem*, to appear in Asterisque.

[17] Yu. Manin and P. Zograf, *Invertible cohomological field theories and Weil-Petersson volumes*, Ann. Inst. Fourier. 50 (2000), 519–535.

[18] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. 167 (2007), 179–222.

[19] M. Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc. 20 (2007), 1–23.

[20] M. Mulase and B. Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy*, Indiana J. Math. 50 (2008), 189–218.
[21] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in *Arithmetic and Geometry* (M. Artin and J. Tate, eds.), Part II, Birkhäuser, 1983, 271-328.
[22] R. Penner, *Weil-Petersson volumes*, J. Differential Geom. 35 (1992), 559–608.
[23] B. Safnuk, *Integration on moduli spaces of stable curves through localization*, math.DG/0704.2530.
[24] G. Schumacher and S. Trapani, *Estimates of Weil- Petersson volumes via effective divisors*, Comm. Math. Phys. 222 (2001), 1–7.
[25] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geometry, vol.1, (1991) 243–310.
[26] S. Wolpert, *On the homology of the moduli space of stable curves*, Ann. Math., 118 (1983) 491–523.
[27] P. Zograf, *An algorithm for computing Weil-Petersson volumes of moduli spaces of curves*, Institut Mittag-Leffler - Preprints 2006/2007.
[28] *A Maple program to compute higher Weil-Petersson volumes*, available at http://www.cms.zju.edu.cn/news.asp?id=1214&ColumnName=pdfbook&Version=english

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